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SEPARABLE COORDINATES FOR THREE-DIMENSIONAL COMPLEX RIEMANNIAN SPACES

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1. Introduction

In this paper we study the problem of separation of variables for the equations

$$(1.1)\quad \begin{aligned} &\text{(a)}\quad \Delta_3\psi = \sum_{i,j=1}^{3} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial \psi}{\partial x^j}\right) = E\psi,\\ &\text{(b)}\quad \sum_{i,j=1}^{3} g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j} = E. \end{aligned}$$

Here $ds^2 = g_{ij}\, dx^i\, dx^j$ is a complex Riemannian metric, $g = \det(g_{ij})$, $g^{ij}g_{jk} = \delta^i_k$, $g_{ij} = g_{ji}$, and E is a nonzero complex constant. Thus (1.1)(a) is the eigenvalue equation for the Laplace-Beltrami operator on a three-dimensional complex Riemannian space whereas (1.1)(b) is the associated Hamilton-Jacobi equation.

We shall classify all metrics for which equations (1.1) admit solutions via separation of variables. Furthermore we shall indicate explicitly the group theoretic significance of each type of variable separation. The separation of variables problem for (1.1) has been studied by other authors, most notably by Stäckel [12], Robertson [11] and Eisenhart [14]. These authors were primarily concerned with systems for which the metric is orthogonal and in Stäckel form. Here, however, we classify all separable systems, orthogonal or not, in Stäckel form or not. Special emphasis is given to the nonorthogonal systems.

It is quite easy to show that (1.1)(b) admits (additive) separation of variables in every coordinate system for which (1.1)(a) admits a product separation and that in general (1.1)(b) separates in more systems than does (1.1)(a). However, we shall prove explicitly that in the cases where g_{ij} corresponds to flat space there is a one-to-one correspondence between separable systems for the equations. In these cases one can pass to Cartesian

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coordinates x, y, z such that $ds^2 = (dx)^2 + (dy)^2 + (dz)^2$ and (1.1) becomes

$$(1.2)\qquad \begin{aligned} &(a) \quad \Delta_3\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = E\psi, \\ &(b) \quad \left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2 = E, \end{aligned}$$

the complex Helmholtz equation and the equation of geometrical optics, respectively. This one-one correspondence also holds for spaces of constant (nonzero) curvature.

Assuming these facts, it follows from earlier papers by the authors [2], [7], [8], [9] that all separable systems in flat spaces or spaces of constant curvature are characterized by pairs of commuting operators which belong to the enveloping algebras of the Lie symmetry algebras of these Riemannian spaces. The symmetry algebras are $\mathcal{E}(3)$, the Lie algebra of the complex Euclidean group, for flat space and o(4) for a space of constant curvature. These Lie algebras are six-dimensional, and all other Riemannian spaces admit symmetry algebras, (the Lie algebras of Killing vectors) whose dimensions are strictly less than six. For general Riemannian metrics such that one of equations (1.1) admits separable solutions it is not necessarily true that the associated pair of symmetry operators belongs to the enveloping algebra of the Lie symmetry algebra. In §5 we shall classify those metrics and coordinates for which the defining symmetry operators do lie in the enveloping algebra. It is in these cases that the separated solutions (special functions) obey addition theorems and recurrence formulas inherited from the symmetry algebra.

As will be shown in §2, aside from some rather elementary cases there is only one class of nonorthogonal separable coordinates for (1.1). All remaining separable coordinates are orthogonal. In §4 we list explicitly the nonorthogonal separable coordinates for flat spaces and spaces of constant curvature. The orthogonal separable coordinates for these spaces have been classified earlier [4], [7], [9], [13].

Havas [6] has also discussed the relationship between the separation of variables problem for (1.1)(a) and (1.1)(b). In particular he gives a condition on the metrics, found by previous authors, which separate the Hamilton-Jacobi equation in order that they also separate the corresponding Helmholtz equation. He also gives a classification of separable coordinate systems for (1.1)(a) which is essentially complete. However, our classification is much simpler, principally because of our definition of equivalence for separable systems. Furthermore, Havas does not exploit the connection between variable separation and symmetry operators.

This connection is exploited by Woodhouse [14] and Dietz [3]. However, these authors do not make a detailed analysis of the possible separable systems for (1.1). Furthermore, Dietz adopts a special definition of separation of variables which omits many well-known separable systems such as ellipsoidal coordinates in flat space.

The operator and group theoretic characterizations of separable systems obtained here are of great importance for the derivation of special function identities [1], [9], [10].

In another paper we shall analyze the separable solutions of (1.1) for the case $E = 0$, where the results are dramatically different.

2. Separable systems for the Helmholtz equation

We begin with the Helmholtz equation (1.1)(a). Our classification is based on the number of ignorable variables which occur when (1.1)(a) is expressed in terms of the separable coordinates x^1, x^2, x^3 under consideration. If x^i is an ignorable coordinate, then in the expression for the Laplace-Beltrami operator Δ_3 the only x^i dependence is through the partial derivative $\partial/\partial x^i$, i.e., $[\partial/\partial x^i, \Delta_3] = 0$, so that $L = \partial/\partial x^i$ is a Lie symmetry of (1.1)(a).

At this point we interrupt our development to recall that the *symmetry algebra* of (1.1)(a) is the Lie algebra $\mathcal{G}$ of all operators

$$L = \sum_{i=1}^{3} \xi^i(x^j)\frac{\partial}{\partial x^i}, \tag{2.1}$$

such that $[L, \Delta_3] = 0$; see, for instance [10]. It is well known [5] that these operators L are exactly the Killing vector fields for the corresponding metric (g_{ij}). Similarly, the second-order symmetries of (1.1)(a) are the second-order differential operators

$$L' = \sum_{i,k} \eta^{ik}(x^j)\frac{\partial^2}{\partial x^i \partial x^k} + \sum_i \xi^i(x^j)\frac{\partial}{\partial x^i}, \tag{2.2}$$

such that $[L', \Delta^3] = 0$. If every such L' agrees with a second-order polynomial in the enveloping algebra of $\mathcal{G}$, then (1.1)(a) is said to be of *class* I [10]. Otherwise (1.1)(a) is of *class* II. We shall demonstrate explicitly that every separable solution $\psi = A(x^1)B(x^2)C(x^3)$ of (1.1)(a) is characterized by a commuting pair of second order symmetries L_1, L_2 such that

$$L_i\psi = \lambda_i\psi, \quad i = 1, 2, \tag{2.3}$$

where the eigenvalues λ_i are the separation constants. In the following we classify each of the separable systems and list the associated operators L_1, L_2.

The first type of metric is that for which all three variables x^i are ignorable. In this case all metric coefficients are constants, $g_{ij} = a_{ij}$, and a typical separable solution of (1.1)(a) has the form $\psi = \exp[C_1x^1 + C_2x^2 + C_3x^3]$. Note, however, that we can always make the change of variables

$$x^i = \sum_{j=1}^{3} \varphi_{ij}(y^j), \quad i = 1, 2, 3, \tag{2.4}$$

which would for essentially arbitrary functions $\varphi_{ij}(y^j)$ produce the separable solution

$$\psi = \exp\left[\sum_{i,j=1}^{3} C_i\varphi_{ij}(y^j)\right] \tag{2.5}$$

in terms of the variables y^i. In our subsequent classification we must bear this degree of freedom in mind, but we do not regard such a change of coordinates as producing a different separable system. In particular we can always take linear combinations of our x^i and get an equivalent set of coordinates. We take as our standard representative the Cartesian coordinates in flat space:

$$\begin{aligned} &[\mathrm{I}] \quad ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \\ &L_1 = \left(\frac{\partial}{\partial x^1}\right)^2, \quad L_2 = \left(\frac{\partial}{\partial x^2}\right)^2. \end{aligned} \tag{2.6}$$

Now suppose there are exactly two ignorable coordinates x^1, x^2. Then the metric coefficients are functions of x^3 alone so that

$$\begin{aligned} &[\mathrm{II}] \quad ds^2 = \sum_{i,j=1}^{3} g_{ij}(x^3)\, dx^i\, dx^j, \\ &L_1 = \frac{\partial}{\partial x^1}, \quad L_2 = \frac{\partial}{\partial x^2}. \end{aligned} \tag{2.7}$$

In the case of one ignorable variable x^1 the Helmholtz equation for the function $\phi(x^2, x^3)$, where $\psi = \exp(C_1x^1)\phi(x^2, x^3)$, has the form

$$\begin{aligned} &a_{22}\phi_{22} + 2a_{23}\phi_{23} + a_{33}\phi_{33} + (2a_{12}C_1 + a_2)\phi_2 \\ &\quad + (2a_{13}C_1 + a_3)\phi_3 + (a_{11}C_1^2 + a_1C_1 - E)\phi = 0, \end{aligned} \tag{2.8}$$

with $a_{ij} = g^{ij}$. There are two possibilities to consider. The separation equations in the variables x^2 and x^3 are either (i) both of second-order or (ii) one is of second-order and the other of first-order. We examine these possibilities in turn.

(i) For both separation equations to be of second-order we must have $a_{23} = g^{23} = 0$, and for suitable choices of x^2, x^3, the components of the contravariant metric tensor take the form

$$g^{11} = Q[R(x^2) + S(x^3)], \quad g^{22} = g^{33} = Q$$
$$g^{12} = QH(x^2), \quad g^{13} = QI(x^3), \quad Q(x^2, x^3) = [U(x^2) + V(x^3)]^{-1}. \tag{2.9}$$

There are further constraints on the metric coefficients in order that (1.1)(a) separates, one of which is $a_2 = (g)^{-1/2}\partial/\partial x^2((g)^{1/2}g^{22}) = Qf(x^2)$. This condition implies

$$\frac{\partial}{\partial x^2}\ln\left[\frac{U+V}{\bar{R}+\bar{S}}\right] = F(x^2) \tag{2.10}$$

where $\bar{R}(x^2) = R - H^2$, $\bar{S}(x^3) = S - I^2$. It follows that

$$\frac{U+V}{\bar{R}+\bar{S}} = J(x^2)K(x^3). \tag{2.11}$$

There are a number of subcases to consider.

(ia) If $U_0 \neq 0$ and $V_3 \neq 0$ then $\bar{R} = \alpha U^{\pm 1}$, $\bar{S} = \alpha V^{\pm 1}$ for some $\alpha \in \mathbf{C}$. This is the only additional requirement for separation. The differential form describing these coordinates is (for the exponent $+1$)

$$[\mathrm{III}] \quad ds^2 = (U(x^2) + V(x^3))\left[(dx^2)^2 + (dx^3)^2\right] + (dX^1)^2$$
$$L_1 = \left(\frac{\partial}{\partial X^1}\right)^2, L_2 = \frac{1}{U+V}\left(V\frac{\partial^2}{\partial(x^2)^2} - U\frac{\partial^2}{\partial(x^3)^2}\right), \tag{2.12}$$

where $\sqrt{\alpha}\, dX^1 = dx^1 - Hdx^2 - Idx^3$. This form illustrates another degree of freedom which we have in separating variables. If there is an ignorable variable x^1 then we can always choose a new ignorable variable X^1 given by

$$x^1 = f(X^1) + g(x^2) + h(x^3),$$

so that X^1, x^2, x^3 will also yield separation. Again we do not regard this as giving a different coordinate system. (We have also ignored variable changes of this sort in the case of type [II] forms).

For the exponent -1 we find similarly

$$[\mathrm{III}]' \quad ds^2 = U(x^2)V(x^3)(dx')^2 + (U+V)\left[U(dx^2)^2 + V(dx^3)^2\right],$$
$$L_1 = \left(\frac{\partial}{\partial x^1}\right)^2, \tag{2.13}$$
$$L_2 = \frac{V-U}{UV}L_1 + \frac{1}{U+V}\left[\frac{V}{U}\left(\frac{\partial}{\partial x^2}\right)^2 - \frac{U}{V}\left(\frac{\partial}{\partial x^3}\right)^2\right].$$

(ib) If $U \equiv 0$ and $\bar{R} \equiv 0$, then $R = H^2$ and $a^2 = 0$. There are no further restrictions on the metric g^{ij}. The differential form can be reduced to

$$ds^2 = P(x^3)(dx^2)^2 + Q(x^3)(dx^1)^2 + (dx^3)^2, \tag{2.14}$$

where $dX^1 = dx^1 - Hdx^2 - \mathrm{I}dx^3$. However, this form is just a particular case of type [II].

(ic) If $U \equiv 0$ and $\bar{S} \equiv 0$, then $S = \mathrm{I}^2$. There are no further restrictions, and the resulting form can be reduced to

$$ds^2 = (dx^3)^2 + V(x^3)\left[(dx^2)^2 + Q(x^2)(dX^1)^2\right] \tag{2.15}$$

with X^1 as in (2.14). By a suitable change of variable $X^2 = F(x^2)$, $X^3 = G(x^3)$ we can write (2.15) in the more convenient form

$$\begin{aligned} &[\mathrm{IV}] \quad ds^2 = V(x^3)(dx^3)^2 + V(x^3)Q(x^2)\left[(dx^2)^2 + (dx^1)^2\right], \\ &L_1 = \frac{\partial^2}{\partial(x^1)^2}, \quad L_2 = \frac{1}{Q}\left(\frac{\partial^2}{\partial(x^1)^2} + \frac{\partial^2}{\partial(x^2)^2}\right), \end{aligned} \tag{2.16}$$

where we have again denoted the variables with lower case letters. Here $V_3 \neq 0$, $Q_2 \neq 0$, for otherwise [IV] becomes a special case of [II].

(ii) For the second possibility we must have $g^{23} = g^{22} = 0$. With a suitable choice of x^2, x^3 the components of the contravariant metric tensor take the form

$$\begin{aligned} &g^{33} = g^{12} = Q, \quad g^{13} = QH(x^3), \\ &g^{11} = Q\left[R(x^2) + S(x^3)\right], \quad Q = \left[U(x^2) + V(x^3)\right]^{-1}. \end{aligned} \tag{2.17}$$

We have the additional conditions $a_2 = 0$ and $a_3 = Qf(x^3)$. This last condition implies

$$\frac{\partial}{\partial x^3}\ln\left[U + V\right] = F(x^3). \tag{2.18}$$

This can only be true if $U_2 = 0$ or $V_3 = 0$. If $U_2 = 0$, then by the change of ignorable variable $x^1 \to X^1$ where $dx^1 = dX^1 + \frac{1}{2}R\,dx^2$ this metric reduces to case [II]. If, however, $V_3 = 0$, then we have the new form

$$\begin{aligned} ds^2 = U(x^2)\Big[&(R(x^2) + S(x^3))(dx^2)^2 \\ &+ (H(x^3)\,dx^2 + dx^3)^2 + 2\,dx^1\,dx^2\Big], \end{aligned} \tag{2.19}$$

or, defining a new ignorable variable $dX^1 = dx + H dx^3 + \frac{1}{2} R\, dx^2$,

$$[\mathrm{V}] \quad ds^2 = A(x^2)\left[B(x^3)(dx^2)^2 + 2dx^1\, dx^2 + (dx^3)^2\right],$$

$$(2.20) \qquad L_1 = \frac{\partial}{\partial x^1}, \quad L_2 = \frac{\partial^2}{\partial (x^2)^2} - B(x^3)\frac{\partial^2}{\partial (x^1)^2}.$$

Note that these coordinates are nonorthogonal and that L_1 is of first-order.

Finally there is the case of separable coordinates in which there are no ignorable variables. Such coordinates must of necessity be orthogonal, and the differential forms are well known to be [4]

$$[\mathrm{VI}] \quad ds^2 = \sigma_1^2(dx^1)^2 + \sigma_1(\sigma_2 + \sigma_3)\left[(dx^2)^2 + (dx^3)^2\right],$$

$$\sigma_i = \sigma_i(x^i), \quad i = 1, 2, 3,$$

$$(2.21) \qquad L_1 = \frac{1}{\sigma_2 + \sigma_3}\left(\frac{\partial^2}{\partial (x^2)^2} + \frac{\partial^2}{\partial (x^3)^2}\right),$$

$$L_2 = \frac{1}{\sigma_2 + \sigma_3}\left(\sigma_3\frac{\partial^2}{\partial (x^2)^2} - \sigma_2\frac{\partial^2}{\partial (x^3)^2}\right);$$

$$[\mathrm{VII}] \quad ds^2 = (q_1 - q_2)(q_1 - q_3)(dx^1)^2 + (q_2 - q_3)(q_2 - q_1)(dx^2)^2 + (q_3 - q_1)(q_3 - q_2)(dx^3)^2, \quad q_i = q_i(x^i), i = 1, 2, 3,$$

$$(2.22) \qquad L_1 = \frac{(q_2 + q_3)}{(q_1 - q_2)(q_3 - q_1)}\frac{\partial^2}{\partial (x^1)^2} + \frac{(q_3 + q_1)}{(q_2 - q_3)(q_1 - q_2)}\frac{\partial^2}{\partial (x^2)^2} + \frac{(q_1 + q_2)}{(q_3 - q_1)(q_2 - q_3)}\frac{\partial^2}{\partial (x^3)^2},$$

$$L_2 = \frac{q_2 q_3}{(q_1 - q_2)(q_3 - q_1)}\frac{\partial^2}{\partial (x^2)^2} + \frac{q_3 q_1}{(q_2 - q_3)(q_1 - q_2)}\frac{\partial^2}{\partial (x^2)^2} + \frac{q_1 q_2}{(q_3 - q_1)(q_2 - q_3)}\frac{\partial^2}{\partial (x^3)^2}.$$

This completes the list of types of differential forms for which the Helmholtz equation (1.1)(a) admits a separation of variables.

3. Separable systems for the Hamilton-Jacobi equation

We now give an analogous classification of coordinate systems for which (1.1)(b) admits variable separation. Recall that separation of variables for

(1.1)(b) means that $W = \sum_{i=1}^{3} W_i(x^i)$ and that the separation equations for the functions W_i are nonlinear equations of second degree and first order.

Following [2] we adopt a phase space formalism to define the symmetry algebra $\mathcal{K}$ of (1.1)(b). The coordinates in this six-dimensional space are (x^j, p_j) where $p_j = \partial W/\partial x^j$. The Poisson bracket of two functions F, G on phase space is the function

$$\{F(x,p), G(x,p)\} = \sum_{j=1}^{3}\left(\frac{\partial G}{\partial x^j}\frac{\partial F}{\partial p_j} - \frac{\partial F}{\partial x^j}\frac{\partial G}{\partial p_j}\right). \tag{3.1}$$

A *first-order symmetry* of (1.1)(b) is a function

$$\mathfrak{L} = \sum_{i=1}^{3} \xi^i(x)p_i, \tag{3.2}$$

such that $\{\mathfrak{L}, \sum_{ij} g^{ij}p_ip_j\} \equiv 0$. It is straightforward to check that the $(\xi^i(x))$ are just the Killing vector fields for the metric (g_{ij}) and that the map $L \to \mathfrak{L}$ is a Lie algebra isomorphism of the Lie symmetry algebra $\mathcal{G}$ of (1.1)(a) (consisting of differential operators (2.1)) and the algebra $\mathcal{K}$ of all first-order symmetries $\mathfrak{L}$ under the Poisson bracket. Moreover, whenever W is a solution of (1.1)(b), then so is $\mathfrak{L}$, (recall $p_j = \partial W/\partial x^j$).

Similarly, the (strictly) *second-order symmetries* of (1.1)(b) are the functions

$$\mathfrak{L} = \sum_{i,j=1}^{3} \eta^{ij}(x)p_ip_j, \quad \eta^{ij} = \eta^{ji}, \tag{3.3}$$

such that $\{\mathfrak{L}, \sum g^{ij}p_ip_j\} = 0$. The vector space of second-order symmetries does not form a Lie algebra but it is decomposed into orbits under the adjoint action of $\mathcal{K}$.

We will show explicitly that every class of separable solutions W of (1.1)(b) is characterized by a pair of first or second-order symmetries $\mathfrak{L}_1$, $\mathfrak{L}_2$ which are in involution: $\{\mathfrak{L}_1, \mathfrak{L}_2\} = 0$. The exact characterization is

$$\mathfrak{L}_1 = \lambda_1, \quad \mathfrak{L}_2 = \lambda_2, \tag{3.4}$$

where λ_1, λ_2 are the separation constants. In the following we classify the separable systems and list the associated functions $\mathfrak{L}_1$, $\mathfrak{L}_2$.

Again the classification is based on the number of ignorable coordinates. In the case of three and two ignorable coordinates the differential forms coincide with [I] and [II] respectively, and the commuting symmetries are $\mathfrak{L}_1 = p_1^2$, $\mathfrak{L}_2 = p_2^2$. If there is one ignorable coordinate x^1 and the separation equations in x^2, x^3 are both of the second degree, then the only restriction on the contravariant metric is that it be of the form (2.9). Thus the differential form can be written as

$$(3.5)\quad \begin{aligned}&(3)\quad ds^2 = \left[U(x^2) + V(x^3)\right]\left[(dx^2)^2 + (dx^3)^2 + \frac{(dX^1)^2}{R(x^2) + S(x^3)}\right],\\ &\mathcal{L}_1 = p_1^2,\quad \mathcal{L}_2 = (U + V)^{-1}\left[(VR - US)p_1^2 + Vp_2^2 - Up_3^2\right],\end{aligned}$$

where $dX^1 = dx^1 - H(x^2)\,dx^2 - I(x^3)\,dx^3$. If the separation equation in the variable x^2 is of first degree, then the only restriction on the contravariant tensor is that it be of the form (2.17). Thus

$$(3.6)\quad \begin{aligned}&(4)\quad ds^2 = \left[U(x^2) + V(x^3)\right]\left[B(x^3)(dx^2)^2 + 2dx^1\,dx^2 + (dx^3)^2\right],\\ &\mathcal{L}_1 = p_1,\quad \mathcal{L}_2 = \left[U + V\right]^{-1}\left[-UBp_1^2 - 2Vp_1p_2 + Up_3^2\right].\end{aligned}$$

Finally, if there are no ignorable variables, then the metric tensor must be diagonal and in Stäckel form [4]. Thus

$$(3.7)\quad \begin{aligned}&(5)\quad ds^2 = \phi\left[\frac{(dx^1)^2}{q_2 - q_3} + \frac{(dx^2)^2}{q_3 - q_1} + \frac{(dx^3)^2}{q_1 - q_2}\right],\\ &\phi = \sum_{i=1}^{3} \varepsilon_{ijk}\phi_i(q_j - q_k),\end{aligned}$$

where $\phi_i = \phi_i(x^i)$, $q_i = q_i(x^i)$ and ε_{ijk} is the completely skew-symmetric tensor such that $\varepsilon_{123} = +1$. Here

$$(3.8)\quad \begin{aligned}\mathcal{L}_1 &= \phi^{-1}\left[(\phi_3 - \phi_2)p_1^2 + (\phi_1 - \phi_3)p_2^2 + (\phi_2 - \phi_1)p_3^2\right],\\ \mathcal{L}_2 &= \phi^{-1}\left[(\phi_2q_3 - \phi_3q_2)p_1^2 + (\phi_3q_1 - \phi_1q_3)p_2^2 + (\phi_1q_2 - \phi_2q_1)p_3^2\right].\end{aligned}$$

This completes the list of forms for which (1.1)(b) admits a separation of variables. It is readily seen that the various types of separation possible for (1.1)(b) are more general than those for the corresponding Helmholtz equation (1.1)(a). (This is to be expected since the separation conditions for (1.1)(b) only impose a general form of contravariant metric tensor whereas the corresponding conditions for (1.1)(a) must also take into account first derivative terms.) We now show, however, that for flat space there is a 1-1 correspondence between separable systems for these equations.

We proceed by examining the consequences of flatness on the possible separable forms for the Hamilton-Jacobi equation. For forms of types (1) and (2) we need make no further comment. The metric of type (3) is in Stäckel form and for flat space is also subject to the condition $R_{ij} = 0$ $(i \neq j)$ where R_{ij} is the Ricci tensor. Eisenhart [4] has shown that this condition together with the requirements that the metric be orthogonal and in Stäckel form imply that the metric (3) also provides a separation of variables for the Helmholtz equation. However, it is easily checked that these conditions force the metric (3) to be in one of the forms [III] or [IV].

To show that the nonorthogonal metric (4) for (1.1)(b) reduces to [V] for the Helmholtz equation, we note that the condition $R_{1221} = \frac{1}{2}[U + V]^{-1}V_3^2 = 0$ on the Riemannian curvature tensor in flat space implies $V_3 = 0$. Thus this metric determines a separation for both (1.1)(a) and (1.1)(b).

Finally, for a differential form of type (5), which is already in Stäckel form, the flat space condition $R_{ij} = 0$ $(i \neq j)$ implies that the metric also separate the Helmholtz equation, hence be of type [VI] or [VII]. We have thus obtained the following result.

Theorem. *Let $x = x(x^i), y = y(x^i), z = z(x^i)$ define a new set of variables $x^i (i = 1, 2, 3)$ in such a way that the coordinates x, y, z are analytic functions of the x^i. This set of new variables admits separable solutions for the Hamilton-Jacobi equation*

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2 = \sum g^{ij} \frac{\partial W}{\partial x^i} \frac{\partial W}{\partial x^j} = E$$

if and only if it also admits separable solutions for the Helmholtz equation $\Delta_3 \psi = E\psi$. Here $ds^2 = dx^2 + dy^2 + dz^2 = \sum g_{ij} dx^i dx^j$ and the separable solutions take the form $W = \sum_{i=1}^3 U_i(x^i), \psi = \Pi_{i=1}^3 V_i(x^i)$.

There is a similar theorem for spaces of (nonzero) constant curvature K_0. (In this case Δ_3 is the Laplace-Beltrami operator on the complex 3-dimensional sphere with radius $K_0^{-1/2}$.) Rather than prove this theorem directly here we will obtain it as a biproduct of results for the Hamilton-Jacobi equation in four space, to be published later.

4. Nonorthogonal separable coordinates in spaces of constant curvature

All orthogonal separable coordinate systems for the real Helmholtz and Klein-Gordon equations

$$(4.1) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Psi = E\Psi, \quad \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\Psi = E\Psi$$

have been classified in [7] and [10], and given a group theoretic interpretation. From this list one can easily obtain a classification of all orthogonal separable systems for the complex Helmholtz equation in flat space, bearing in mind that distinct real systems may be complex equivalent. Here we will compute the possible nonorthogonal separable systems in flat space.

Recall that the symmetry algebra of the flat space Helmholtz equation is $\mathcal{E}(3)$ with basis (in Cartesian coordinates)

$$J_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad J_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad J_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$

(4.2)

$$P_1 = \frac{\partial}{\partial x}, \quad P_2 = \frac{\partial}{\partial y}, \quad P_3 = \frac{\partial}{\partial z}.$$

This equation is of class I, so all separable systems are describable in terms of commuting operators L_1, L_2 in the enveloping algebra of $\mathcal{E}(3)$. The possible nonorthogonal systems can only be of types [II] or [V].

We begin by classifying the possible forms of type [V]. The requirement

$$R_{3223} = \frac{1}{2}AB'' + \frac{1}{2}A'' - \frac{3}{4}\frac{(A')^2}{A} = 0 \tag{4.3}$$

for the Riemann curvature tensor, where A and B are given by (2.20), leads to the separation equations

$$B'' = \alpha, A'' + \alpha A - \frac{3}{2}\frac{(A')^2}{A} = 0.$$

If $\alpha \neq 0$, these equations have the solution

$$B = \frac{\alpha}{2}(x^3)^2 + \beta x^3 + \gamma,$$

$$A = \delta \sec^2\left(\sqrt{\frac{-\alpha}{2}}\; x^2\right) \text{ or } A = \delta \exp(\sqrt{2\alpha}\; x^2).$$

By suitable redefinition of variables, these solutions yield the two metrics

(a) $ds^2 = x^2(dx^3)^2 + 2\,dx^1\,dx^2 + \frac{1}{4}\frac{(x^3)^2}{x^2}(dx^2)^2,$

(b) $ds^2 = \left[1 + (x^2)^2\right](dx^3)^2 + 2\,dx^1\,dx^2 - \frac{(x^3)^2}{\left[1 + (x^2)^2\right]}(dx^2)^2.$

If $\alpha = 0$ we have the solutions

$$B = \beta x^3 + \gamma, \quad A = (\delta x^2 + \mathcal{E})^{-2},$$

which determine the form

(c) $ds^2 = (x^2)^2(dx^3)^2 + 2\,dx^1\,dx^2 - \frac{4ax^3}{(x^2)^2}(dx^2)^2.$

These are the only possibilities for systems of type [V]. We now give the coordinate systems corresponding to these possibilities, the operators which describe them, and the separation equations for the Helmholtz equation. (Here $\{L_1, L_2\} = L_1L_2 + L_2L_1$.)

(a) $x = x^3(x^2)^{1/2}, \quad y - iz = x^1 - \frac{1}{4}(x^3)^2, \quad y + iz = 2x^2,$

$$\Delta_3\psi = \left\{\frac{1}{x^2}\left[\left(\frac{\partial}{\partial x^3}\right)^2 - \frac{1}{4}\left(x^3\frac{\partial}{\partial x^1}\right)^2\right] + 2\frac{\partial^2}{\partial x^1 \partial x^2}\right\}\psi = E\psi,$$

$$L_1 = \frac{\partial}{\partial x^1} = \frac{1}{2}\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial z}\right) = \frac{1}{2}(P_2 + iP_3),$$

$$L_2 = \left(\frac{\partial}{\partial x^3}\right)^2 - \frac{1}{4}\left(x^3\frac{\partial}{\partial x^1}\right)^2 = \frac{1}{2}\{P_1, J_3 - iJ_2\}.$$

(b) $x = x^3\sqrt{1 + (x^2)^2}\,, \quad y - iz = x^1 - \frac{1}{2}(x^3)^2x^2, \ y + iz = 2x^2,$

$$\Delta_3\psi = \left\{(1 + (x^2)^2)^{-2}\left[\left(\frac{\partial}{\partial x^3}\right)^2 + \left(x^3\frac{\partial}{\partial x^1}\right)^2\right] + 2\frac{\partial^2}{\partial x^1 \partial x^2}\right\}\psi = E\psi,$$

$$L_1 = \frac{1}{2}(P_2 + iP_3), \quad L_2 = P_1^2 + \frac{1}{4}(J_3 - iJ_2)^2.$$

(c) $x = x^3x^2 + \frac{a}{x^2}, \quad y - iz = x^1 - \frac{1}{2}(x^3)^2x^2 + \frac{ax^3}{x^2} + \frac{a^2}{6(x^2)^3},$

$$y + iz = 2x^2,$$

$$\Delta_3\psi = \left\{(x^2)^{-2}\left[\left(\frac{\partial}{\partial x^3}\right)^2 + 4ax^3\left(\frac{\partial}{\partial x^1}\right)^2\right] + 2\frac{\partial^2}{\partial x^2 \partial x^2}\right\}\psi = E\psi,$$

$$L_1 = \frac{1}{2}(P_2 + iP_3), \quad L_2 = \frac{1}{4}(J_3 - iJ_2)^2 + 2aP_1(P_2 + iP_3).$$

We see that these three coordinate systems (as well as system (4.5)) correspond to the imbedding of the heat equation into three-dimensional space via the change of coordinates

$$x = x^3, y - iz = x^1, y + iz = 2x^2,$$

(4.4)
$$ds^2 = (dx^3)^2 + 2\,dx^1\,dx^2,$$

$$\Delta_3\psi = \left(\left(\frac{\partial}{\partial x^2}\right)^2 + 2\frac{\partial^2}{\partial x^1 \partial x^2}\right)\psi = E\psi.$$

Diagonalization of the operator $L_1 = \frac{1}{2}(P_2 + iP_3) = \partial/\partial x^1$ reduces the Helmholtz equation to the heat equation.

The only remaining possibilities for nonorthogonal separable metrics are those of type [II]. However, type [II] systems, which we have referred to as *split* in reference [2], can be classified in terms of orbits of pairs of commuting symmetry operators under the adjoint action of $\mathcal{E}(3)$. It is easy to verify that, except for the orbits of linear momentum operator pairs which correspond to type [I] (Cartesian) coordinates, there are only two kinds of orbits. The first orbit contains the representative J_3, P_3 and corresponds to cylindrical coordinates (which are orthogonal) while the second kind of orbit contains $P_2 + iP_3, J_2 + iJ_3 + c(P_2 - iP_3)$ where $c = 0, 1$. Here

$$(4.5)\quad \begin{aligned} x &= ic(x^2)^2 + x^2x^3, \\ y &= cx^2 + \frac{c}{3}(x^2)^3 + x^1 - \frac{i}{2}(x^2)^2x^3, \\ z &= -icx^2 + \frac{ic}{3}(x^2)^3 + ix^1 + \frac{1}{2}(x^2)^2x^3 - x^3, \end{aligned}$$

$$ds^2 = (x^3)^2(dx^2)^2 + (dx^3)^2 + 2ic\, dx^2\, dx^3 + 4c\, dx^1\, dx^2 - 2i\, dx^1\, dx^3.$$

This completes our list of nonorthogonal separable coordinates for the flat space Helmholtz equation.

We next consider the possible separable systems for spaces of constant curvature K_0. The 21 possible orthogonal separable systems were classified in [9]. The nonorthogonal separable systems are of types [V] or [II]. For systems of type [V] we must require that the curvature tensor of the metric satisfy the condition

$$(4.6)\qquad R_{hijk} = K_o(g_{hj}g_{ik} - g_{hk}g_{ij}),\ K_0 \neq 0.$$

A tedious but straightforward computation shows that there are in fact no metrics of type [V] satisfying (4.6). Thus the only possibilities for nonorthogonal separable metrics are those of type [II]. Now the metrics of type [II] can be classified in terms of orbits of pairs of commuting symmetry operators under the adjoint action of O(4), the symmetry group of (1)(a) for spaces of constant curvature. Recall that the space of constant curvature K_0 can be locally identified with the complex sphere S_{3c}: $z_1^2 + z_2^2 + z_3^2 + z_4^2 = K_0^{-1}$. The symmetry algebra of this space is o(4) with basis

$$(4.7)\qquad I_{jk} = z_j\frac{\partial}{\partial z_k} - z_k\frac{\partial}{\partial z_j},\quad 1 \leqslant j, k \leqslant 4,\ j \neq k.$$

Under the adjoint action of the group o(4) there are exactly three orbits with representatives (I_{23}, I_{14}), $(I_{42} + iI_{21}, I_{34} + iI_{13})$ and $(I_{21} + I_{43}, I_{12} + I_{43} + iI_{14} + iI_{32})$, respectively. The first two orbits correspond to orthogonal coordinates and are listed in [9]. The third system is nonorthogonal with metric

$$(4.8)\quad \begin{aligned} ds^2 &= K_0^{-1/2}\left[(dx^2)^2 - (dx^3)^2 - 2e^{2x^3}\,dx^1\,dx^2\right],\\ z_1K_0^{1/2} &= x^1e^{x^3}\sin x^2 + \cosh x^3\cos x^2,\\ z_2K_0^{1/2} &= x^1e^{x^3}\cos x^2 - \cosh x^3\sin x^2,\\ z_3K_0^{1/2} &= ix^1e^{x^3}\sin x^2 - i\sinh x^3\cos x^2,\\ z_4K_0^{1/2} &= ix^1e^{x^3}\cos x^2 + i\sinh x^3\sin x^2. \end{aligned}$$

This is the only nonorthogonal separable system for spaces of constant curvature.

5. Class I coordinates

As mentioned above, the Helmholtz equations for flat space and spaces of constant curvature are of class I, so the defining operators L_1, L_2 for each separable coordinate system associated with these equations belong to the enveloping algebra of the Lie symmetry algebra. For such systems one can then employ the representation theory of $\mathcal{E}(3)$ and o(4) to obtain significant properties of the separable solutions [9], [10], [1]. However, this happy state of affairs does not hold for general Riemannian spaces.

To explore the relationship between Lie symmetries and separable systems more carefully we introduce a new definition. A separable coordinate system $\{x^1, x^2, x^3\}$ for equation (1.1)(a) is of *class I* if the defining operators L_1, L_2 for this system belong to the enveloping algebra of the Lie symmetry algebra for (1.1)(a). Otherwise, the coordinate system is *class* II. (Note that a class II equation may still have a separable coordinate system of class I.)

It is obvious that all separable systems corresponding to type [I] and [II] metrics are of class I. For type [III] metrics we have the following result.

Theorem. *Let* $\{x^1, x^2, x^3\}$ *be a class* I *separable system of type* [III]. *Then the metric is one of two forms*:

(a) $ds^2 = (dx)^2 + (dy)^2 + (dz)^2$, *symmetry algebra* $\mathcal{E}(3)$,

(b) $ds^2 = (dx^1)^2 + d\omega^2$ *where* $d\omega^2$ *is the metric for a two-dimensional Riemannian space of nonzero constant curvature, symmetry algebra* $\mathbf{C} \times$ o(3).

We sketch the proof of this theorem. It is easy to see that L_2 cannot be expressed as a polynomial in $\partial/\partial x^1$ and only one first-order symmetry operator. Thus the system $\{x^j\}$ can be of class I only if (2.12) admits a symmetry algebra $\mathcal{G}$ which is at least three-dimensional. From the Killing equations for the metric (2.12) (i.e., the requirements on the operator L, (2.1) such that $[L, \Delta_3] = 0$) and the integrability conditions for these equations, one obtains

$$A_1 = 0, A_2 = -(U + V)B_1,$$
$$A_3 = -(U + V)C_1, C_2 = -B_3,$$
$$B_2 = C_3 = -(U + V)^{-1}(BU_2 + CV_3),$$
$$A_2R = A_3R = 0, R = U_{22} + V_{33} - \frac{(U_2^2 + V_3^2)}{U + V}.$$

Furthermore, writing $ds^2 = (dx^1)^2 + d\omega^2$, one can easily verify that the curvature of the metric $d\omega^2$ is given by $R_{1221} = \frac{1}{2}R$. If $R = 0$, then ds^2 is the metric of a flat space and $\{x^j\}$ is clearly of class I. If $R \neq 0$, then A is a constant and $B_1 = C_1 = 0$. Thus the symmetry algebra decomposes as $\mathbf{C} \times \mathcal{G}'$ where $\mathcal{G}'$ is the symmetry algebra of $d\omega^2$ and dim $\mathcal{G}' \geqslant 2$. Now it is well-known [5, p. 243] that a two-dimensional Riemannian space with symmetry algebra of dimension $\geqslant 2$ must either be flat space (which we have already counted above) or a space of nonzero constant curvature. This completes the proof.

Theorem. *Let $\{x^1, x^2, x^3\}$ be a class* I *separable system of type* [4]. *Then the metric is one of four forms*:

(a) *flat space, symmetry algebra* $\mathcal{E}(3)$, *where* $(1/V)'' = 0$, $(Q^{-1/2})'' + \lambda Q^{-1/2} = 0$,

(b) *space of nonzero constant curvature, symmetry algebra* o(4), $(1/V)'' = a \neq 0$, $(Q^{-1/2})'' + \lambda Q^{-1/2} = 0$,

(c) $ds^2 = V^2(dx^3)^2 + V\, d\omega^2(x^1, x^2)$ *where* $d\omega^2$ *is the flat space metric, symmetry algebra* $\mathcal{E}(2)$,

(d) $ds^2 = V^2(dx^3)^2 + V\, d\omega^2(x^1, x^2)$ *where* $d\omega^2$ *is the metric for a space of constant curvature, symmetry algebra* o(3).

The proof of this result is similar to that of the preceding theorem.

Theorem. *A class* I *separable system of type* [V] *corresponds to a flat-space metric and a class* I *system of type* [III]′ *corresponds to a flat space or constant curvature metric. A class* I *separable system of type* [VI] *corresponds to one of four metric forms*:

(a) *flat space, symmetry algebra* $\mathcal{E}(3)$, *where*

$$\sigma_1 = c(x^1)^{-2},$$
$$\Phi = \frac{2}{\sigma_1(\sigma_2 + \sigma_3)}\left[\frac{\sigma_2'' + \sigma_3''}{\sigma_2 + \sigma_3} - \frac{(\sigma_2')^2 + (\sigma_3')^2}{(\sigma_2 + \sigma_3)^2}\right] + \frac{(\sigma_1')^2}{\sigma_1^4} = 0,$$

(b) *space of constant curvature* K_0, *symmetry algebra* o(4), *where*

$$\sigma_1 = (cx^2 + bx + a)^{-1}, K_0 = b^2 - 4ac, \Phi = -4K_0,$$

(c) $ds^2 = \sigma_1^2(dx^1)^2 + \sigma_1\, d\omega^2(x^2, x^3)$ *where* $d\omega^2$ *is the flat-space metric, symmetry algebra* $\mathfrak{E}(2)$,

(d) $ds^2 = \sigma_1^2(dx^1)^2 + \sigma_1\, d\omega^2(x^2, x^3)$ *where* $d\omega^2$ *is the metric for a space of constant curvature, symmetry algebra* o(3).

We have not yet been able to determine the class I separable systems for metrics of type [VII], although it appears likely that flat and constant curvature metrics are the only ones possible.

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